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TURBULENT ISENTROPIC FLOWS

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Based the hypothesis that there is a direct proportional relationship between turbulent stresses and pairwise products of the averaged velocity components, isentropic flows are studied.

In a number of the works of the present author (for example, [1]), in order to express the turbulent stress tensor $(-\rho \overline{v_j' v_k'})$ in terms of the averaged velocity field, a hypothesis is suggested according to which turbulent stresses are proportional to pairwise products of the average velocity components of turbulent motion:

$$-\rho \overline{v_j' v_k'} = \beta \rho \bar{v}_j \bar{v}_k \,, \quad j, k = 1, 2, 3 \,, \tag{1}$$

where the parameter β characterizes the intensity of pulsating motions.

If, in the well-known Reynolds equations, we use formulas (1) for averaged velocities, then for an incompressible fluid we will have

$$\frac{\partial \overline{\mathbf{V}}}{\partial t} + (1 - \beta)(\overline{\mathbf{V}} \cdot \nabla) \overline{\mathbf{V}} = -\frac{1}{\rho} \operatorname{grad} p + \nu_0 \nabla^2 \overline{\mathbf{V}} \,. \tag{2}$$

Here $\overline{\mathbf{V}}$ is the averaged motion velocity vector; ν_0 is the effective viscosity, with the remaining notation being conventional. In [2, 3], calculating formulas are given for β and ν_0 in application to some classes of hydrodynamic flows.

Let us denote the speed of a liquid particle moving along a streamline by V_s. According to the definition of the streamline, we have the obvious formulas

$$\frac{dx}{dS} = \frac{\bar{u}}{V_s} = l \,; \quad \frac{dy}{dS} = \frac{\bar{v}}{V_s} = m \,; \quad \frac{dz}{dS} = \frac{\bar{w}}{V_s} = n \,;$$

$$V_s^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2 \,, \quad V_s = l\bar{u} + m\bar{v} + n\bar{w} \,.$$

Let us take note of the formulas of conversion from the derivatives with respect to x, y, z to the variable S:

$$\frac{\partial}{\partial x} = \frac{1}{l} \frac{\partial}{\partial S} \,; \quad \frac{\partial}{\partial y} = \frac{1}{m} \frac{\partial}{\partial S} \,; \quad \frac{\partial}{\partial z} = \frac{1}{n} \frac{\partial}{\partial S} \,;$$

$$\frac{d}{dS} = l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \,.$$

In the new variables S and t, using the well-known procedures, Eq. (2) can be rewritten as

$$\frac{d}{dS} \left(\frac{p}{\rho} + (1 - \beta) \frac{V_s^2}{2} \right) = -\nu_0 A \frac{\partial^2 V_s}{\partial S^2} - \frac{\partial V_s}{\partial t} \,, \tag{3}$$

where the complex A has the following form:

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$$A = 6 - \left(\frac{l^4 + n^4}{l^2 n^2} + \frac{l^4 + m^4}{l^2 m^2} + \frac{m^4 + n^4}{m^2 n^2} \right).$$

If $l = m = n$, then $A = 0$, and Eq. (3) is simplified to

$$\frac{d}{dS} \left(\frac{p}{\rho} + \frac{1 - \beta}{2} V_s^2 \right) = - \frac{\partial V_s}{\partial t}.$$

Hereafter, we will consider steady-state motions of liquid or gas, for which the latter equation has the integral

$$p + \frac{1 - \beta}{2} \rho V_s^2 = H. \tag{4}$$

In this equation, the quantity H defines the total pressure of each elementary small jet in a given active section of the flow. To obtain the total pressure $\overline{H}$, which is the hydrodynamic characteristic of the entire active section ω, it is necessary to take averages of the values of H that belong to separate small jets. Then, instead of Eq. (4), we obtain the equality

$$\frac{1}{Q} \int_\omega \left(p + \frac{1 - \beta}{2} \rho V_s^2 \right) dQ = \frac{1}{Q} \int_\omega H dQ.$$

By integrating all the terms of this equation over the active section plane ω, we will have a Bernoulli equation generalized for the case of turbulent flows:

$$\frac{p}{\rho} + \frac{\alpha (1 - \beta)}{2} W^2 = \overline{H}, \tag{5}$$

where the flow rate of the hydrodynamic stream is denoted by W.

In hydraulics, the quantity α is called the correction of the kinetic energy of flow, and for its determination the following formula is valid:

$$\alpha = \frac{1}{\omega W^3} \int_\omega V_s^3 \, d\omega.$$

Equation (5) at $\beta = 0$ was already used in solving hydraulics problems. Thus, it is known that in laminar flows, in transition from slow to accelerated motions, the numerical values of α change from 2 to 0.5; at the same time, in turbulent flows the value of α is close to unity. The introduction of the parameter β into Eq. (5) expands the capabilities of this equation in describing the conservation law in turbulent flows.

For a thermally insulated flow, with the gas state equation in the Clapeyron–Mendeleev form being satisfied, Eq. (5) becomes

$$\frac{k}{k - 1} \frac{p}{\rho} + \alpha (1 - \beta) \frac{W^2}{2} = \frac{k}{k - 1} \frac{p_0}{\rho_0}. \tag{6}$$

In Eq. (6) the subscript "0" pertains to the retarded-gas flow parameters, while k denotes the ratio of heat capacities. The gasdynamic processes will be considered to be isentropic. For these processes, the relationship between pressure and density $p = c\rho^k$ is valid whereas the speed of sound is determined as

$$a = \sqrt{\left(k \frac{p}{\rho} \right)} = \sqrt{kRT}.$$

This allows one to put Eq. (6) in the form

$$\frac{a^2}{k-1} + \alpha(1-\beta)\frac{W^2}{2} = \frac{a_0^2}{k-1}. \qquad (7)$$

The ratio between the gas-flow velocity W and the local speed of sound is called the Mach number M. Then, the energy equation (7) can be represented as

$$\frac{T_0}{T} = 1 + \frac{\alpha(1-\beta)(k-1)}{2}\mathrm{M}^2. \qquad (8)$$

The mean flow velocity W, equal to the speed of sound, will be called the mean critical velocity W_{cr}. From Eq. (7) it is easy to see that

$$W_{cr}^2 = \frac{2}{2+\alpha(1-\beta)(k-1)}a_0^2.$$

From Eq. (6) we can easily obtain

$$\frac{p_0}{p} = \left[1 + \frac{\alpha(1-\beta)(k-1)}{2}\mathrm{M}^2\right]^{\frac{k}{k-1}}$$

Now we assume that M = *1* and $p = p_{cr}$, after which the latter relationship becomes

$$\frac{p_{cr}}{p_0} = \left[\frac{2}{2+\alpha(1-\beta)(k-1)}\right]^{\frac{k}{k-1}}. \qquad (9)$$

In turbulent flow of liquid in a tube, the complex $\alpha(1-\beta)$ depends little on the molecular viscosity-based Reynolds number, and is calculated as [2]

$$\alpha(1-\beta) = 0.45036 + 4.32\cdot 10^{-8}\,\mathrm{Re}.$$

In accordance with this relationship, $0.4504 \le \alpha(1-\beta) \le 0.5368$ if $4\cdot 10^3 \le \mathrm{Re} \le 2\cdot 10^6$, which for air in turn means $0.7394 \le p_{cr}/p_0 \le 0.6998$.

The complex $\alpha(1-\beta)$ and the ratio p_{cr}/p_0 have still wider ranges of change at the places where the flow either sharply expands or contracts. In [3], where local resistances of the hydrodynamic flow were studied on the basis of Eq. (5), the following empirical relation was obtained for a sudden flow expansion:

$$\alpha(1-\beta) = 2.6135 - 1.4891\frac{\omega_1}{\omega_2}. \qquad (10)$$

Here ω_1 and ω_2 denote the cross-sectional areas of the narrow and expanded portions of the flow, respectively. It is also shown there that on sudden contraction of the flow

$$\alpha(1-\beta) = 1.4075 - 0.8913\frac{\omega_2}{\omega_1}. \qquad (11)$$

Formulas (10) and (11) are also applicable to air flows. Table 1 illustrates the ranges of variation of the ratio p_{cr}/p_0 in air flows where the indicated local resistances occur.

Now we will consider the problem of gas escape from a reservoir through a nozzle, when heat exchange with the environment can be neglected due to the short dwell time of the gas in the nozzle. Let the pressure in the interior of the reservoir be equal to p_0, the density to ρ_0, and the gas velocity in the reservoir be neglected. The gas parameters at the nozzle exit will not be subscripted. We will make use of Eq. (6) and solve it for the flow velocity:

TABLE 1. The Nature of Change in the Critical Pressure in Flows with Local Resistance

Flow expansion			Flow contraction		
ω_1/ω_2	$\alpha(1-\beta)$ from Eq. (10)	p_{cr}/p_0	ω_1/ω_2	$\alpha(1-\beta)$ from Eq. (11)	p_{cr}/p_0
0.9	1.2733	0.4520	0.9	0.6053	0.6703
0.8	1.4222	0.4164	0.8	0.6945	0.6343
0.7	1.5711	0.3843	0.7	0.7836	0.6008
0.6	1.7200	0.3553	0.6	0.8727	0.5695
0.5	1.8689	0.3291	0.5	0.9618	0.5402
0.4	2.0179	0.3053	0.4	1.0510	0.5129
0.3	2.1668	0.2836	0.3	1.1401	0.4873
0.2	2.3157	0.2639	0.2	1.2292	0.4633
0.1	2.4646	0.2460	0.1	1.3184	0.4408

$$W = \sqrt{\left(\frac{2k}{\alpha(1-\beta)(k-1)} \frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}\right]\right)}. \tag{12}$$

For the steady-state gas stream its mass flow rate is $Q = \rho\omega W$ or, with allowance for Eq. (12),

$$Q = W\sqrt{\left(\frac{2kp_0\rho_0}{\alpha(1-\beta)(k-1)}\left[\left(\frac{p}{p_0}\right)^{2/k} - \left(\frac{p}{p_0}\right)^{\frac{k+1}{k}}\right]\right)}. \tag{13}$$

In Eq. (13), ω denotes the active cross-sectional area of the flow. If, in Eq. (13), we assume that $\alpha(1-\beta) = 1$, we obtain the well-known St.Venant–Wentzel formula. However, formula (13) has a more general character, since it contains a correction for the complex $\alpha(1-\beta)$ that characterizes the local resistances in nozzles.

To determine the maximum flow rate, we will take the derivative of the right-hand side of Eq. (13) with respect to the variable p/p_0 and equate it to zero. Then, after simple transformations, we obtain

$$\frac{p_m}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{14}$$

Here p_m denotes the limiting pressure at the nozzle exit. Comparing Eqs. (9) and (14), we note that $p_{cr} \neq p_m$, contrary to the generally accepted concepts. In order to obtain the velocity W_m, it is necessary to replace the quantity p/p_0 in Eq. (12) by p_m/p_0 defined by formula (14). Then we have

$$W_m = \frac{1}{\sqrt{\alpha(1-\beta)}} a_m,$$

where a_m is the speed of sound at the pressure p_m. From this it follows that in isentropic turbulent escape of gas from a reservoir the maximum flow rate occurs at an escape velocity not equal to the speed of sound.

In the same manner, Eq. (13) also yields a formula also for the maximum flow rate:

$$Q_m = \omega\sqrt{\left(\frac{kp_0\rho_0}{\alpha(1-\beta)}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\right)}. \tag{15}$$

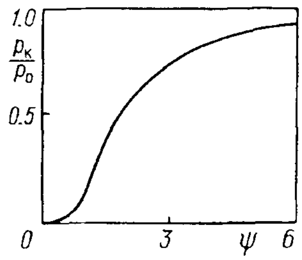


Fig. 1. Change in the critical pressure on escape of gas from a reservoir.

In practice in gasdynamic calculations, when the St.Venant–Wentzel formula is fitted to experimental data, an empirical multiplier μ is used, called the flow-rate coefficient. This leads to the following formula, suitable for calculating the flow rate of a gas on its escape from the reservoir:

$$Q = \mu\omega \sqrt{\left(\frac{kp_0\rho_0}{k-1}\left[\left(\frac{p}{p_0}\right)^{2/k} - \left(\frac{p}{p_0}\right)^{\frac{k+1}{k}}\right]\right)}. \tag{16}$$

The disadvantages of this formula are that the multiplier μ is determined only experimentally and that it depends on both the escape conditions and design features of the nozzle.

Comparing Eqs. (13) and (16), we obtain

$$\mu = \frac{1}{\sqrt{\alpha(1-\beta)}} = \psi\sqrt{\left(\frac{k-1}{k}\right)}. \tag{17}$$

Here the quantity ψ is introduced instead of μ for the convenience of further calculations. In [4], it is shown that for the design of the nozzle selected the flow-rate coefficient is not a constant value, but is a function of the flow cross section of the nozzle, i.e., in the process of controlling escape conditions, μ changes from zero to infinity. The formulas suggested earlier for the complex $\alpha(1-\beta)$ can be considered as computational schemes for the empirical multiplier μ in Eq. (16) depending on the character of hydrodynamic flows.

Using Eq. (17), formula (9) can be rewritten as

$$\frac{p_{cr}}{p_0} = \left[\frac{2\psi^2}{k+2\psi^2}\right]^{\frac{k}{k-1}}. \tag{18}$$

If $\alpha(1-\beta) = 1$, then $p_{cr} = p_m$; then from Eq. (14) for air ($k = 1.4$) we have $p_m/p_0 = p_{cr}/p_0 = 0.5283$. The indicated value of p_{cr}/p_0 can be obtained from Eq. (18) if we assume that $\psi = 1.8709$. Figure 1 illustrates the behavior of the function $p_{cr}/p_0 = f(\psi)$ calculated according to Eq. (18).

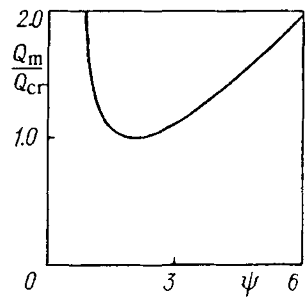
Now we determine the magnitude of the critical flow rate Q_{cr} by substituting Eq. (9) into Eq. (13), after which we have

$$Q_{cr} = \omega\sqrt{\left(kp_0\rho_0\left[\frac{2}{2+\alpha(1-\beta)(k-1)}\right]^{\frac{k+1}{k-1}}\right)}. \tag{19}$$

It can be easily shown that $Q_{cr} = Q_m$ when $\psi = 1.8709$.

There is one other interesting relation:

$$\frac{Q_m}{Q_{cr}} = \frac{\sqrt{\left(\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\right)}}{\sqrt{\left(\frac{k}{(k-1)\psi^2}\left[\frac{2\psi^2}{2\psi^2+k}\right]^{\frac{k+1}{k-1}}\right)}}. \tag{20}$$



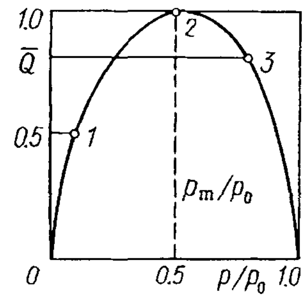


Fig. 2. Flow-rate characteristic.

Fig. 3. Different conditions of gas escape.

From Fig. 2, it follows that for air flows the relation $Q_m/Q_{cr} = f(\psi)$ has a minimum at $\psi = 1.8709$.

Usually, theoretical calculations operate on the dimensionless flow rate $\overline{Q} = Q/Q_m$. A formula for calculating this flow rate can easily be obtained from Eqs. (13) and (15). It has the form

$$\overline{Q} = \sqrt{\left(\frac{2}{k-1}\left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}}\left[\left(\frac{p}{p_0}\right)^{2/k} - \left(\frac{p}{p_0}\right)^{\frac{k+1}{k}}\right]\right)}. \tag{21}$$

Relation (21) determines the character of air escape from a reservoir depending on the pressure ratio p/p_0 (Fig. 3). In Fig. 3, we will note point 2, where $p/p_0 = p_m/p_0 = 0.5283$, $\overline{Q} = 1$, $Q/Q_{cr} = 1$, and $\psi = 1.8709$. Under these conditions, when $p/p_0 < p_m/p_0$, gas escapes with the maximum flow rate Q_m up to point 2, while downstream of it ($p/p_0 > p_m/p_0$) the flow rate decreases with an increase in the values of p/p_0. This is a well-known character of escape, but within the framework of the concept stated, this is a particular case, which occurs only at $\psi = 1.8709$.

For further analysis we use the formula

$$\frac{W_m}{W_{cr}} = \sqrt{\left(\frac{(k-1)(k+2\psi^2)}{k(k+1)}\right)}, \tag{22}$$

which determines the ratio of the maximum velocity of escape W_m to the critical velocity W_{cr}.

Now, we will consider the case where the nozzle has the generalized flow-rate coefficient $\psi < 1.8709$. Let $\psi = 1$; then from Eq. (18) we have $p_{cr}/p_0 = 0.1561$. We plot this value on the abscissa axis (Fig. 3) and through this point we draw a straight line parallel to the ordinate axis up to the intersection with the curve $\overline{Q} = f(p/p_0)$. The point of intersection is assigned number 1. When $\psi = 1$, from Eq. (22) we obtain that the maximum velocity of escape W_m is smaller than the critical one W_{cr}. Therefore, upstream to point 1 ($0 < p/p_0 \le 0.1561$), the flow rate is equal to the critical one, and thereafter ($0.1561 < p/p_0 \le 1$) it increases up to point 2 and then decreases to zero.

The third regime of escape occurs when $\psi > 1.8709$. Suppose that $\psi = 4$, which corresponds to $p_{cr}/p_0 = 0.8608$ (point 3 in Fig. 3). In this case, from Eq. (22) we have $W_m > W_{cr}$. Consequently, the flow rate of escape is equal to the critical one up to point 3 ($0 < p/p_0 \le 0.8608$), and then it decreases to zero.

Thus, in application to turbulent isentropic flows, a new regime of escape from point 2 to point 3 (Fig. 3) is determined, which was not dealt with earlier in the available literature on turbulent flows.

REFERENCES

1. V. A. Bubnov, in: Hydromechanics and Elasticity Theory, No. 32 [in Russian], Dnepropetrovsk (1984), pp. 29-36.
2. V. A. Bubnov, Vestsi Akad. Navuk BSSR, Ser. Fiz.-Énerg. Navuk, No. 1, 121-122 (1990).
3. V. A. Bubnov, Vestn. Mashinostr., No. 11, 17-20 (1989).
4. V. A. Bubnov, Vestn. Mashinostr., No. 12, 34-36 (1989).